

# MATH 2050C Lecture 3 (Jan 19)

[[ Take-home Midterm confirmed: Mar 11, 6pm - Mar 12, 6pm. ]]

Goal:  $\mathbb{R}$  is a complete ordered field.  
*today*

Def<sup>n</sup>: (Absolute value) Let  $a \in \mathbb{R}$ .

$$|a| := \begin{cases} a & \text{if } a > 0 \\ 0 & \text{if } a = 0 \\ -a & \text{if } a < 0 \end{cases}$$

Note:  $|a| \geq 0 \quad \forall a \in \mathbb{R}$

Prop: (a)  $|ab| = |a| \cdot |b|$

(b)  $|a|^2 = a^2$

\* (c) Let  $c \geq 0$ . Then  $|a| \leq c \iff -c \leq a \leq c$

(d)  $-|a| \leq a \leq |a|$

Proof: (a) We exhaust all possible cases from Trichotomy (02).

Case 1: Either  $a$  or  $b$  is 0.

Then,  $ab = 0 \implies |ab| = 0$ .

Also, if  $a = 0$ , then  $|a| = 0 \implies |a| \cdot |b| = 0$ .

Same for  $b = 0$ . So,  $|ab| = |a| |b| = 0$ .

Case 2:  $a > 0$  and  $b < 0$ .

Then, by Prop. last time,  $ab < 0 \implies |ab| = -ab$ .

Also,  $a > 0 \implies |a| = a$

$b < 0 \implies |b| = -b$

$$\left. \begin{array}{l} a > 0 \implies |a| = a \\ b < 0 \implies |b| = -b \end{array} \right\} |a| \cdot |b| = a \cdot (-b) = -ab$$

// Same

Ex: Check this!

Case 3:  $a > 0$  and  $b > 0$

Case 4:  $a < 0$  and  $b < 0$

Case 5:  $a < 0$  and  $b > 0$

Ex:

Same as Case 2

(b) Take  $b = a$  in (a),

$$a^2 = |a^2| = |ab| = |a||b| = |a| \cdot |a| = |a|^2.$$

$$\uparrow \therefore a^2 \geq 0 \quad \forall a \in \mathbb{R}.$$

(c) Exhaust all cases of  $a$  by trichotomy (Ex:)

(d) Follows from (c) by taking  $C = |a| \geq 0$ . \_\_\_\_\_ ◻

### Some Useful Inequalities

(1) AM-GM inequality:  $\sqrt{ab} \geq 0 \leq \frac{1}{2}(a+b) \quad \forall a, b \geq 0$

(2) Triangle inequality:  $|a+b| \leq |a| + |b| \quad \forall a, b \in \mathbb{R}$

(3) Bernoulli's inequality:  $(1+x)^n \geq 1+n \cdot x \quad \forall x > -1, \forall n \in \mathbb{N}$

Proof: (1) Let  $a, b \geq 0$ , then  $\sqrt{a}, \sqrt{b}$  exist (Assume this).

By previous lemma,

$$\begin{aligned} 0 &\leq (\sqrt{a} - \sqrt{b})^2 = (\sqrt{a})^2 - 2\sqrt{a}\sqrt{b} + (\sqrt{b})^2 \\ &= a - 2\sqrt{a}\sqrt{b} + b \end{aligned}$$

Rearranging gives the desired inequality.

(2) By (d) above, we have

$$\left. \begin{array}{l} -|a| \leq a \leq |a| \\ -|b| \leq b \leq |b| \end{array} \right\} \begin{array}{l} \xrightarrow{\text{add}} \\ \xrightarrow{(c)} \end{array} \begin{array}{l} -( |a| + |b| ) \leq a + b \leq |a| + |b| \\ |a + b| \leq |a| + |b|. \end{array}$$

(3) Induction on  $n$ .

$n=1$ : Trivial since  $(1+x)^1 = 1+x = 1+n \cdot x$ , when  $n=1$ .

Assume  $n=k$  is true, then for  $n=k+1$ ,

$$(1+x)^{k+1} = (1+x)(1+x)^k$$

$\forall x > -1 \quad \forall n=k$

$$\begin{aligned}
&\geq (1+x)(1+k \cdot x) && \left( \begin{array}{l} \because n=k \text{ is true} \\ \text{and } x > -1 \end{array} \right) \\
&= 1 + (k+1)x + k \cdot x^2 \\
&\geq 1 + (k+1)x && (\because k > 0, x^2 \geq 0)
\end{aligned}$$

By M.I., we are done. \_\_\_\_\_ ◻

Remark: Let  $a, b \geq 0$ . Then

$$a \leq b \iff a^2 \leq b^2 \iff \sqrt{a} \leq \sqrt{b}.$$

Prop: (Reversed Triangle Ineq.)

$$||a| - |b|| \leq |a - b| \quad \forall a, b \in \mathbb{R}.$$

Pf: Tutorial.

Def<sup>n</sup> / Thm (Completeness Property of  $\mathbb{R}$ )

Every  $\emptyset \neq S \subseteq \mathbb{R}$  that has an "upper bound" must have a "supremum" in  $\mathbb{R}$ .

We first make sense of the '?'s.

Def<sup>n</sup>: Let  $\emptyset \neq S \subseteq \mathbb{R}$ .

(a)  $S$  is bounded above if  $\exists u \in \mathbb{R}$  s.t.  $s \leq u \quad \forall s \in S$   
Any such  $u \in \mathbb{R}$  is called an upper bound of  $S$ .

(b)  $S$  is bounded below if  $\exists w \in \mathbb{R}$  s.t.  $s \geq w \quad \forall s \in S$   
Any such  $w \in \mathbb{R}$  is called an lower bound of  $S$ .

(c)  $S$  is bounded if it is both bdd above AND below.

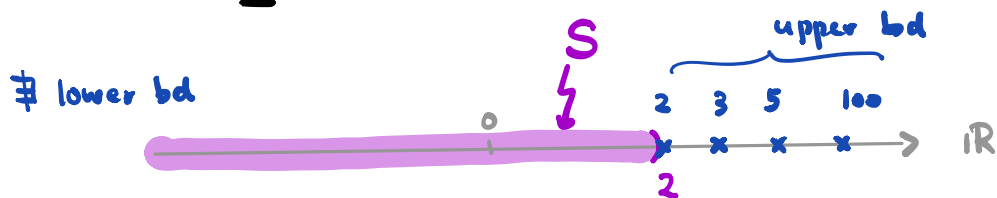
Otherwise,  $S$  is unbounded.

Example:  $S := \{x \in \mathbb{R} \mid x < 2\}$

Note: There are many upper bds, e.g. 2, 3, 5, 100,  $\sqrt{100}$  etc...

$\Rightarrow S$  is bdd above.

BUT  $S$  is NOT bdd below. (Ex: prove it)



Def<sup>n</sup>: Let  $\emptyset \neq S \subseteq \mathbb{R}$ .

(a) Suppose  $S$  is bdd above.

Then,  $u \in \mathbb{R}$  is called a **supremum** (or **least upper bound**) of  $S$  if the following holds:

(i)  $u$  is an upper bd of  $S$

(ii)  $u \leq v$  for any upper bd  $v$  of  $S$

[ Notation:  
 $u = \sup S$  or l.u.b.  $S$  ]

(b) Similarly, we can define **infimum** (or **greatest lower bound**)

[ Notation:  $\inf S$  or g.l.b.  $S$  ] Ex: Write this down.

Lemma:  $\sup S$ , if exists, is unique.

Proof: Suppose there are two  $u, w \in \mathbb{R}$  which are supremum of  $S$

Therefore,  $u, w$  satisfy (i) . (ii) in the def<sup>n</sup> above.

By (i) for  $w$  and (ii) for  $u$ , we have

$$u \leq w \quad \checkmark \quad \because w \text{ is an upper bd}$$

Similarly, by (i) for  $u$  and (ii) for  $w$ , we have

$$w \leq u \leftarrow \because u \text{ is an upper bd.}$$

Thus,  $u = w$ .

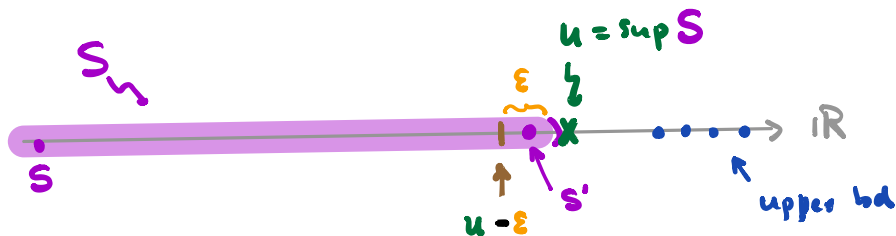
Prop: Let  $\emptyset \neq S \subseteq \mathbb{R}$ . Then  $u = \sup S$  iff

(i)  $s \leq u \quad \forall s \in S$

(ii)  $\forall \epsilon > 0, \exists s' \in S \text{ s.t. } u - \epsilon < s'$

Useful way to prove  $u = \sup S$

Picture:



Proof: " $\Rightarrow$ " Suppose  $u = \sup S$ .

By (i),  $u$  is an upper bd of  $S$

$$\Rightarrow u \geq s \quad \forall s \in S \quad \text{which is (i).}$$

By (ii),  $u \leq v$  for any upper bd.  $v$  of  $S$ . (\*)

Fix  $\epsilon > 0$ , but arbitrary.

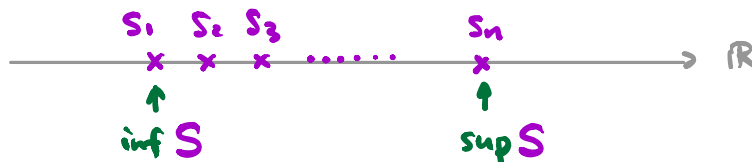
Since  $u - \epsilon < u$ , (\*)  $\Rightarrow u - \epsilon$  cannot be an upper bd.

So,  $\exists s' \in S \text{ s.t. } u - \epsilon < s'$

" $\Leftarrow$ " Exercise.

Examples:

1)  $S = \{s_1, \dots, s_n\}$  "finite set" (Assume:  $s_1 < s_2 < \dots < s_n$ )

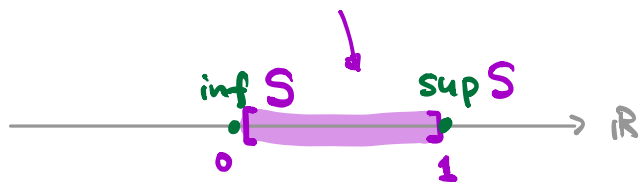


$$\sup S = s_n$$

$$\inf S = s_1$$

(Ex: Prove.)

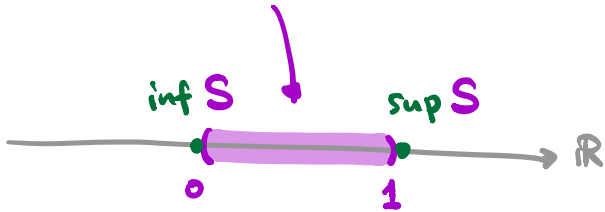
$$2) S = [0, 1]$$



$$\sup S = 1 \in S$$

$$\inf S = 0 \in S$$

$$3) S = (0, 1)$$



$$\sup S = 1 \notin S$$

$$\inf S = 0 \notin S$$